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A Model of Conformal Kinematics

ANATOL ODZIJEWICZ

Institute of Theoretical Physics, Warszawa, Poland

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Abstract

A new realization of the conformal group in the classical phase space of a positive mass relativistic scalar particle is obtained. A physical interpretation in the spirit of conformal relativity is discussed.

1. *Introduction*

The conformal group has recently become an object of attention in several branches of theoretical physics. By conformal transformations are understood the elements of the full conformal group G_{conf} including the nonlinear Hanties transformations ("accelerations") as well as the traditional Poincaré transformations. The conformal group is essential in relativistic cosmology. It independently appears in quantum mechanics where it describes the symmetries of the hydrogen atom. However, the most essential role of the conformal group is due to the fact that it forms the invariance group of the charge-free Maxwell electrodynamics. This motivates the use of G_{conf} in elementary particle physics as a "skeleton symmetry group" describing the asymptotic behavior of particle systems. Mathematically, the G_{conf} is a simple Lie group locally isomorphic to both matrix groups $O(2.4)$ and $SU(2.2)$. The Lie algebra of G_{conf} is the smallest simple Lie algebra containing the algebra of the Poincaré group.

The conformal transformations, in general, do not conserve the space-time metric, but they do conserve the light cone. This might lead to an idea about a conformal relativity being a generalized version of special relativity (see, e.g., Page, 1936). A possible step in this direction is considered in this paper, where a new realization of G_{conf} as a group of operations on a classical phase space is constructed. In the first part of the paper some elementary concepts concerning the symplectic manifolds are quoted and the mechanism that makes the symplectic manifolds the homogeneous spaces of arbitrary Lie groups is outlined. In the second part this mechanism is used to provide the realization of G_{conf} on a generalized complex disc. Then, a mapping of the generalized disc onto a symplectic manifold of the positive mass relativistic scalar particle is constructed. In the last section the possible physical sense of the resulting representation is discussed and its asymptotic correspondence to the standard idea of the classical phase space is proved.

The following standard notation will be used:

$T(P)$ – tangent bundle
$T^*(P)$ – cotangent bundle

$$T^{(k,l)}(P) = [\overset{k}{\underset{i=1}{\otimes}} T(P)] \otimes [\overset{l}{\underset{j=1}{\otimes}} T^*(P)]$$

$\overset{k}{\wedge} T(P)$, $\overset{k}{\wedge} T^*(P)$ – exterior products of $T(P)$ and $T^*(P)$, respectively
$\Gamma(\mathscr{E}, U)$ – set of sections of the bundle $\mathscr{E}$ over U
$\Gamma(T^{(k,l)}(P))$ – set of tensor fields on P
$A(\Omega)$ – set of holomorphis functions on $\Omega \subset \mathbb{C}^1$
$\mathscr{L}$ – Lie derivative
$\lrcorner$ – left interior product
$\wedge$ – exterior product
$\odot$ – symmetrization

2. *Symplectic Manifolds as Homogeneous Spaces of Arbitrary Lie Groups*

The notion of symplectic manifold generalizes the physical idea of phase space, where the "Poisson bracket" is a fundamental structure.

> *Definition.* A symplectic manifold is any pair (P, ω) where (1) P is a $2n$-dimensional differential manifold; (2) $\omega \in \Gamma(\overset{2}{\wedge} T^*(P))$, $d\omega = 0$; and (3) the 2-form ω is nonsingular.

Briefly, the symplectic manifold is an even-dimensional differential manifold with a closed nonsingular 2-form chosen. Given a symplectic manifold (P, ω) the group $D(P, \omega)$ of canonical transformations is defined as the automorphism group of (P, ω); that is, $D(P, \omega) = \{\tau \in D(P)\colon \ \overset{2}{\wedge}\tau^*\omega = \omega\}$, $D(P)$ being the group of all diffeomorphism $P \leftrightarrow P$. The existence of a distinguished 2-form ω on P induces two mappings that intercommunicate the 1-forms and vector fields on P:

$$(1) \quad \omega_\flat : \Gamma(T(P)) \to \Gamma(T^*(P))$$

$$(2) \quad \omega_\# \overset{df}{=} \omega_\flat^{-1} : \Gamma(T^*(P)) \to \Gamma(T(P))$$

The mapping $\omega_\flat$ is defined by $\omega_\flat(\xi) = \xi \lrcorner \omega$, where $\xi \in \Gamma(T(P))$ and the $\omega_\#$ is the unique mapping inverse to $\omega_\flat$ (the existence of $\omega_\#$ follows from the nonsingularity of ω). A vector field $\xi \in \Gamma(T(P))$ is called locally Hamiltonian if $\underset{\xi}{\mathscr{L}}\omega = 0$. The locally Hamiltonian fields form a linear space, which will be denoted by $H(P)$. Obviously $H(P)$ is also a Lie algebra with respect to the commutation of the vector fields. Since

$$\underset{\xi}{\mathscr{L}}\omega = \xi \rfloor d\omega + d(\xi \rfloor \omega) = d[\omega_\flat(\xi)] \tag{2.1}$$

then the image of $H(P)$ under the mapping $\omega_\flat$ is contained in the set $B\Gamma(T^*(P))$ of all closed 1-forms on P. A vector field is called globally Hamiltonian if there exists a $\varphi \in C^\infty(P, R^1)$ such that $\omega_\flat(\xi) + d\varphi = 0$. The set of globally Hamiltonian fields $H_0(P)$ forms an ideal in $H(P)$. The quotient Lie algebra $H(P)/H_0(P)$ is isomorphic to the first de Rham group of cohomology of the manifold $P: H^1(P, R^1)$. The form ω introduces the structure of an infinite-dimensional Lie algebra on the function space $C^\infty(P, R^1)$.

Definition. Let $f, g \in C^\infty(P, R^1)$ and let $\gamma = \omega_\# \circ d : C^\infty(P, R^1) \to H_0(P)$. Then the skew product (Poisson bracket) is defined by

$$\{f, g\} \overset{df}{=} -\omega(\gamma(f), \gamma(g)) = \gamma(f)g = -\gamma(g)f \tag{2.2}$$

The product $\{\ ,\ \}$ is a skew, bilinear form satisfying the Jacobi identity. It is introduced to correspond to the commutator in the Lie algebra of vector fields on P. The correspondence is given by the mapping γ. In fact, γ is an homomorphism of the algebra $C^\infty(P, R)$ into the algebra $H_0(P)$:

$$[\gamma(f), \gamma(g)] = \gamma(\{f, g\}) \tag{2.3}$$

The homomorphism kernel is the set of all real functions constant on P. This might be summarized in the form of the following exact sequence of Lie algebras:

$$0 \longrightarrow R^1 \xrightarrow{\iota} C(P, R^1) \xrightarrow{\gamma} H_0(P) \longrightarrow 0 \tag{2.4}$$

where ι is the natural injection.

In many problems of mathematical physics the symplectic manifold appears as a realization space for the elements of a Lie group (which is then interpreted as the symmetry group of a physical system). This leads to the following notion of a symplectic manifold being a homogeneous space of a Lie group G:

Definition. Given a Lie group G with a Lie algebra $\mathscr{G}$, a G-symplectic space is (P, ω, G, σ), where (1) (P, ω) form a symplectic manifold, and (2) σ is an homomorphism of G into the group of automorphism $D(P, \omega)$ such that $\underset{p_1, p_2 \in P}{\bigwedge} \underset{g \in G}{\bigvee} p_2 = \sigma(g)p_1$.

The group homomorphism σ induces the homomorphism of the Lie algebras $d\sigma : \mathscr{G} \to H(P)$ defined by

$$[d\sigma(x)f](p) = \left.\frac{d}{dt}\right|_{t=0} f[\sigma(\exp(-tx))p], \qquad x \in \mathscr{G}, f \in C^\infty(P, R^1) \tag{2.5}$$

The homogeneous space P can be naturally identified with any of quotient spaces G/G_p, where G_p is the small group (stabilizer) of any point $p \in P$. The G-space P is called a strictly homogeneous symplectic space of the group G if $d\sigma(\mathscr{G}) \subset H_0(P)$, i.e., if the following diagram commutes:

$$\begin{tikzcd} & & & 0 \arrow[d] & \\ 0 \arrow[r] & R_1 \arrow[r, "\iota"] & C^\infty(P, R^1) \arrow[r, "\gamma"] & H_0(P) \arrow[r] \arrow[d] & 0 \\ & & \mathscr{G} \arrow[u, "\lambda"'] \arrow[r, "d\sigma"] & H(P) \arrow[d] & \\ & & & H^1(P, R^1) \arrow[d] & \\ & & & 0 & \end{tikzcd} \tag{2.6}$$

Here, λ is an homomorphism which is called the uprised $d\sigma$.

3. *Geometrical Structures Naturally Connected with the Generalized Disc D*

We begin this section with an outline of the geometrical structures generated by the complex structure of a bounded single connected domain M in the complex n-plain $\mathbb{C}^n$. The injection ι: $M \subset \mathbb{C}^n$ defines the natural coordinate system on the whole, bounded domain: $\iota(z) = (z^1, \ldots, z^n, \bar{z}^1, \ldots, \bar{z}^n)$ for $z \in M$.

The space

$$\mathscr{H}_M = \left\{ f \in \mathscr{A}\Gamma(\overset{n}{\wedge} T^*(M)) \colon \int_M i^{n^2} f \wedge \bar{f} < \infty, f = f^*\, dz^1 \wedge \ldots \wedge dz^n, f^* \in \mathscr{A}(M) \right\}$$

of the holomorphic n-forms f that are square integrable is a complex Hilbert space, where the scalar product is given by

$$(f, g) \overset{df}{=} \int_M i^{n^2} f \wedge \bar{g}$$

($\bar{f}$ is the antiholomorphic form $\bar{f}^*\, d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n$).

The geometric Riemannian and symplectic structures may be defined on M by introducing the Bergman kernel form.

Let $\{h_0, h_1, \ldots\}$ be a complete orthonormal basis in $\mathscr{H}_M$. the form

$$K \overset{df}{=} K^*(z, \bar{z})\, dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n = \sum_{i=0} h_i \wedge \bar{h}_i \tag{3.1}$$

is precisely the Bergman kernel of the bounded domain M. This definition of K does not depend on the choice of the orthonormal basis in $\mathscr{H}_M$. In the new system of coordinates $z = z(v, \bar{v})$, $\bar{z} = \bar{z}(v, \bar{v})$, $[z = (z^1, \ldots, z^n)$, $\bar{z} = (\bar{z}^1, \ldots, \bar{z}^n)$

are understood as independent variables] the Bergman kernel function $K^*(z, \bar{z})$ takes the form

$$K^*(v, \bar{v}) = K^*(z, \bar{z}) \det \frac{\partial(z, \bar{z})}{\partial(v, \bar{v})} \tag{3.2}$$

If the coordinate transformation is given by the holomorphic mapping $z = z(v)$, then

$$\det \frac{\partial(z, \bar{z})}{\partial(v, \bar{v})} = \left| \frac{\partial z}{\partial v} \right|^2$$

The form

$$H = \sum_{k,m=1}^{n} \frac{\partial^2 \log K^*(z, \bar{z})}{\partial z^k \partial \bar{z}^m} dz^k \otimes d\bar{z}^m$$

defines a Hermitian structure on M. H is said to be Hermitian if the following conditions are fulfilled:

$$\begin{aligned} &(1)\ H(\lambda_1 \xi_1 + \lambda_2 \xi, \eta) = \lambda_1 H(\xi_1, \eta) + \lambda_2 H(\xi_2, \eta) \\ &(2)\ \overline{H(\xi, \eta)} = H(\eta, \xi) \\ &(3)\ H(i\xi, \eta) = iH(\xi, \eta) \end{aligned} \tag{3.3}$$

where $\xi_1, \xi_2, \eta \in \Gamma(T(M))$. Now the condition (2) is fulfilled because of the equation

$$\overline{g_{k\bar{m}}} = g_{m\bar{k}}$$

where

$$g_{k\bar{m}} = \frac{\partial^2 \log K^*(z, \bar{z})}{\partial z^k \partial \bar{z}^m}.$$

As easily seen, the remaining conditions are fulfilled too.

In view of these facts, one can now introduce the Riemannian and symplectic structure of the bounded domain M. We have $H = ds^2 + i\omega$, where $ds^2 = \mathrm{Re}H$ and $\omega = \mathrm{Im}H$. From the definition of H we have the following:

(a) ds^2 is the real symmetric 2-form and it is also proved to be positive defined and nonsingular. So, ds^2 defines some metric structure on M, which is called the Riemann-Bergman metric structure:

$$ds^2 = \sum_{k,m=1}^{n} g_{k\bar{m}}\, dz^k \odot d\bar{z}^m \tag{3.4}$$

($\odot$ denotes the symmetrization).

(b) The imaginary part ω is a nonsingular real skew 2-form; moreover, $d\omega = 0$, hence (M, ω) is the symplectic manifold:

$$\omega = -i \sum_{k,m=1}^{n} g_{k\bar{m}}\, dz^k \wedge d\bar{z}^m \tag{3.5}$$

Both geometrical structures are uniquely defined by the complex structure of M.

We now consider a subgroup of all those diffeomorphisms $M \leftrightarrow M$, which preserve the complex structure of M. Evidently, $G(M)$ is the group of holomorphic transformations of M onto itself. Since ds^2 and ω are fully determined by the complex structure of M, $G(M)$ is a group of isometries of the Riemannian manifold (M, ds^2) and also a subgroup of canonical transformations of the symplectic manifold (M, ω). In the theory of differential complex manifolds it is proved (see Kobayashi, 1972), that the group $G(M)$ is a Lie group with a compact isotropic subgroup $G_x(M)$, $x \in M$. We now apply that procedure to our concrete case, with M being a generalized complex four-dimensional disc.

Let $\mathrm{Mat}_2(\mathbb{C}^1)$ be the complex 2 x 2 matrix algebra. An arbitrary matrix $z \in \mathrm{Mat}_2(\mathbb{C}^1)$ may be written in the form $z = z^\mu \sigma_\mu$ where $z^\mu \in \mathbb{C}^1$ and $\sigma_0 = E = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_2 = \left(\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right)$, $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The Hermitian matrix $E - zz^+$ can be transformed with the help of a unitary transformation to the diagonal form $E - zz^+ = U\left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)U^+$, where $U \in U(2)$ and λ_1, λ_2 are eigenvalues. We write $E - zz^+ > 0$ if $\lambda_1 > 0$ and $\lambda_2 > 0$.

Definition. The generalized disc is the following subset of $\mathrm{Mat}_2(\mathbb{C}^1)$:

$$D \overset{df}{=} \{z \in \mathrm{Mat}_2(\mathbb{C}^1) : E - zz^+ > 0\}$$

$$= \{z \in \mathrm{Mat}_2(\mathbb{C}^1) : \mathrm{Tr} zz^+ < 2 \text{ and } \det (E - zz^+) > 0\}$$

As immediately seen, D is an open subset of the ball with radius 2 and thus is bounded. The injection $\iota : D \subset \mathrm{Mat}_2(\mathbb{C}^1) \cong \mathbb{C}^4$ defines a complex coordinate system on D : $\iota(z) = (z^0, z^1, z^2, z^3)$. Taking the homogeneous normalized polynomials in the variables z^μ to be a basis in the Hilbert space $\mathscr{H}_D$, we obtain the Bergman kernel form for the generalized disc:

$$K(z, z^+) = \tilde{c}[\det(E - z^+z)]^{-4}\, dz^0 \wedge \cdots \wedge dz^3 \wedge d\bar{z}^0 \wedge \ldots \wedge d\bar{z}^3 \qquad (3.6)$$

$$K^*(z^\mu, \bar{z}^\nu) = c[1 - 2z^\mu \bar{z}^\nu \delta_{\mu\nu} + (z)^2(\bar{z})^2]^{-4}$$

$$z^+ = \bar{z}^\mu \sigma_\mu, \qquad \delta_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix},$$

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \qquad \tilde{c}, c = \text{const}$$

$$(z)^2 = \eta_{\mu\nu} z^\mu z^\nu \tag{3.7}$$

The coordinates of the Hermitian form H in the basis $(dz^\lambda, d\bar{z}^\gamma)$ are

$$g_{\lambda\bar{\gamma}} = 8[\det(E - z^+ z)]^{-2}[(z_\lambda(\bar{z})^2 - z^\varphi \delta_{\varphi\lambda})(\bar{z}_\gamma(z)^2 - z^\rho \delta_{\rho\varphi}) - \det(E - z^+ z)\bar{z}_\gamma z_\lambda] \tag{3.8}$$

The group $G(D)$ of holomorphic transformations of the generalized disc onto itself is isomorphic to the group $SU(2, 2)/z_4$ where $z_4 = (id, -id, i(id), -i(id))$.

The group $SU(2, 2)$ is one of matrix realizations of the conformal group preserving the form $z^+\eta z$, where $z \in \mathbb{C}^4$ and $\eta = \left(\begin{smallmatrix} E & 0 \\ 0 & -E \end{smallmatrix}\right)$. If we put $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ $\alpha, \beta, \gamma, \delta \in \mathrm{Mat}_2(\mathbb{C}^1)$ then g belongs to $SU(2, 2)$ provided that the following conditions are fulfilled:

$$\begin{aligned} &\alpha^+\alpha - \gamma^+\gamma = E \\ &\alpha^+\beta - \gamma^+\delta = 0 \\ &\beta^+\beta - \delta^+\delta = -E \qquad \det g = 1 \end{aligned} \tag{3.9}$$

Isomorphism $\sigma : SU(2, 2)/z^2 \to G(D)$ is given by

$$z^1 = \sigma(g)z = (\alpha z + \beta)(\gamma z + \delta)^{-1} \tag{3.10}$$

The above definition is meaningful for $z \in D$ and $\alpha, \beta, \gamma, \delta$ fulfilling (3.9).

We have proved that D is the homogeneous space of $SU(2, 2)$, so we can put $D \cong SU(2, 2)/S^1 \times SU(2) \times SU(2)$, where $S^1 \times SU(2) \times SU(2)$ is the stabilizer of the point $z = 0 \in D$.

We are mostly interested in the symplectic structure (D, ω); in what follows we shall describe some of its fundamental properties. D is a simply connected domain. Hence the first group of cohomology is trivial, i.e., $H^1(D, R^1) \cong 0$, and the locally Hamiltonian fields on D are globally Hamiltonian. Thus, diagram (2.6) may be simplified to the form

$$\begin{tikzcd} 0 \arrow[r] & R^1 \arrow[r, "\iota"] & C^\infty(D, R^1) \arrow[r, "\gamma"] \arrow[d, "\lambda_D"'] & H_0(D) \arrow[r] & 0 \\ & & \mathscr{SU}(2, 2) \arrow[ur, "d\sigma_D"'] & & \end{tikzcd} \tag{3.11}$$

We shall find now the values of the homomorphisms $d\sigma_D$ and λ_D. With this aim we specify a basis in the Lie algebra $\mathcal{SU}(2,2)$:

$$e_{re}^{ik} = \begin{pmatrix} 0 & \epsilon^{ik} \\ \epsilon^{ki} & 0 \end{pmatrix}, \qquad e_{im}^{nm} = \begin{pmatrix} 0 & i\epsilon^{nm} \\ -i\epsilon^{mn} & 0 \end{pmatrix}, \qquad e_r = \begin{pmatrix} i\sigma_r & 0 \\ 0 & 0 \end{pmatrix}$$

$$k_l = \begin{pmatrix} 0 & 0 \\ 0 & i\sigma_l \end{pmatrix}, \qquad f_0 = i \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} \tag{3.12}$$

where $k, l, m, n = 1, 2, r, s = 1, 2, 3$ and $\epsilon^{11} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\epsilon^{12} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $\epsilon^{21} = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $\epsilon^{22} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Because of (2.5) the values of $d\sigma_D$ in that basis are

$$d\sigma_D(e_{re}^{11}) = \frac{1}{2}\left[(z^0 + z^3)z^\mu \frac{\partial}{\partial z^\mu} - \frac{\partial}{\partial z^0} - \frac{\partial}{\partial z^3} - (z)^2\left(\frac{\partial}{\partial z^0} - \frac{\partial}{\partial z^3}\right)\right] + (\text{conjugate term}) \tag{3.13a}$$

$$d\sigma_D(e_{re}^{12}) = \frac{1}{2}\left[(z^1 - iz^2)z^\mu \frac{\partial}{\partial z^\mu} - \frac{\partial}{\partial z^1} + i\frac{\partial}{\partial z^2} + (z)^2\left(\frac{\partial}{\partial z^1} + i\frac{\partial}{\partial z^2}\right)\right] + (*) \tag{3.13b}$$

$$d\sigma_D(e_{re}^{21}) = \frac{1}{2}\left[(z^1 - iz^2)z^\mu \frac{\partial}{\partial z^\mu} - \frac{\partial}{\partial z^1} - i\frac{\partial}{\partial z^2} + (z)^2\left(\frac{\partial}{\partial z^1} - i\frac{\partial}{\partial z^2}\right)\right] + (*) \tag{3.13c}$$

$$d\sigma_D(e_{re}^{22}) = \frac{1}{2}\left[(z^0 - z^3)z^\mu \frac{\partial}{\partial z^\mu} - \frac{\partial}{\partial z^0} + \frac{\partial}{\partial z^3} - (z)^2\left(\frac{\partial}{\partial z^0} + \frac{\partial}{\partial z^3}\right)\right] + (*) \tag{3.13d}$$

$$d\sigma_D(e_{im}^{11}) = \frac{-i}{2}\left[(z^0 + z^3)z^\mu \frac{\partial}{\partial z^\mu} + \frac{\partial}{\partial z^0} + \frac{\partial}{\partial z^3} - (z)^2\left(\frac{\partial}{\partial z^0} - \frac{\partial}{\partial z^3}\right)\right] + (*) \tag{3.13e}$$

$$d\sigma_D(e_{im}^{12}) = \frac{-i}{2}\left[(z^1 - iz^2)z^\mu \frac{\partial}{\partial z^\mu} + \frac{\partial}{\partial z^1} + i\frac{\partial}{\partial z^2} + (z)^2\left(\frac{\partial}{\partial z^1} + i\frac{\partial}{\partial z^2}\right)\right] + (*) \tag{3.13f}$$

$$d\sigma_D(e_{im}^{21}) = \frac{-i}{2}\left[(z^1 - iz^2)z^\mu \frac{\partial}{\partial z^\mu} + \frac{\partial}{\partial z^1} + i\frac{\partial}{\partial z^2} + (z)^2\left(\frac{\partial}{\partial z^1} - i\frac{\partial}{\partial z^2}\right)\right] + (*) \tag{3.13g}$$

$$d\sigma_D(e_{im}^{22}) = \frac{-i}{2}\left[(z^0 - z^3)z^\mu \frac{\partial}{\partial z^\mu} + \frac{\partial}{\partial z^0} - \frac{\partial}{\partial z^3} - (z)^2\left(\frac{\partial}{\partial z^0} + \frac{\partial}{\partial z^3}\right)\right] + (*) \tag{3.13h}$$

$$d\sigma_D(e_1) = -i\left[z^1 \frac{\partial}{\partial z^0} - iz^2 \frac{\partial}{\partial z^3} + z^0 \frac{\partial}{\partial z^1} + iz^3 \frac{\partial}{\partial z^2}\right] + (*) \tag{3.13i}$$

$$d\sigma_D(e_2) = -iz^2 \frac{\partial}{\partial z^0} + z^1 \frac{\partial}{\partial z^3} - z^3 \frac{\partial}{\partial z^1} - iz^0 \frac{\partial}{\partial z^2} + (*) \tag{3.13j}$$

$$d\sigma_D(e_3) = -i\left[z^3 \frac{\partial}{\partial z^0} + z^0 \frac{\partial}{\partial z^3} + iz^2 \frac{\partial}{\partial z^1} - iz^1 \frac{\partial}{\partial z^2}\right] + (*) \tag{3.13k}$$

$$d\sigma_D(k_1) = -i\left[-z^1 \frac{\partial}{\partial z^0} - iz^2 \frac{\partial}{\partial z^3} - z^0 \frac{\partial}{\partial z^1} + iz^3 \frac{\partial}{\partial z^2}\right] + (*) \tag{3.13l}$$

$$d\sigma_D(k_2) = iz^2 \frac{\partial}{\partial z^0} + z^1 \frac{\partial}{\partial z^3} - z^3 \frac{\partial}{\partial z^1} + iz^0 \frac{\partial}{\partial z^2} + (*) \tag{3.13m}$$

$$d\sigma_D(k_3) = -i\left[-z^3 \frac{\partial}{\partial z^0} - z^0 \frac{\partial}{\partial z^3} + iz^2 \frac{\partial}{\partial z^1} - iz^1 \frac{\partial}{\partial z^2}\right] + (*) \tag{3.13n}$$

$$d\sigma_D(f_0) = -iz^\mu \frac{\partial}{\partial z^\mu} + i\bar{z}^\mu \frac{\partial}{\partial \bar{z}^\mu} \tag{3.13o}$$

The values of the lifting λ_D are:

$$\lambda_D(e_-^{ik}) = -4i \operatorname{Tr} e^{ik}[z^+(E - zz^+)^{-1} + (E - z^+z)^{-1}z^+] \tag{3.14a}$$

$$\lambda_D(e_+^{ik}) = -4i \operatorname{Tr} \epsilon^{ki}[z(E - z^+z)^{-1} - (E - zz^+)^{-1} z] \tag{3.14b}$$

$$\lambda_D(e_r) = 4 \operatorname{Tr} \sigma_r[(E - zz^+)^{-1} + z(E - z^+z)^{-1}z^+] \tag{3.14c}$$

$$\lambda_D(k_s) = -4 \operatorname{Tr} \sigma_s[z^+(E - zz^+)^{-1}z + (E - z^+z)^{-1}] \tag{3.14d}$$

$$\lambda_D(f_0) = 4 \operatorname{Tr}[(E - zz^+)^{-1} + (E - z^+z)^{-1} + z(E - z^+z)^{-1}z^+ + z^+(E - zz^+)^{-1}z] \tag{3.14e}$$

We have introduced the notation: $e_-^{ik} = \frac{1}{2}(e_{re}^{ik} - ie_{im}^{ik})$ and $e_+^{ik} = \frac{1}{2}(e_{re}^{ik} + ie_{im}^{ik})$.

4. *Symplectic Space $T^*(M_4)$*

Now we shall briefly discuss the basic properties of the conventional phase space of the relativistic particle. Let M_4 be the Minkowski space and (q^0, q^1, q^2, q^3) the coordinate system on M_4. The cotangent bundle $T^*(M_4)$ with its canonical projection $\Pi : T^*(M_4) \to M_4$ is the space for a relativistic particle. The 1-forms dq^0, dq^1, dq^2, dq^3 generated by the coordinates are a basis, and an arbitrary $u \in T^*(M_4)$ can be expressed by

$$u = \sum_{\mu=0}^{3} p_\mu(q)\, dq^\mu \tag{4.1}$$

Hence, (q^μ, p_ν) is a coordinate system on $T^*(M_4)$. The closed 2-form

$$\sigma = d\Pi^* u = \sum_{u=0}^{3} dp_\mu \wedge dq^\mu$$

determines the symplectic manifold structure on $T^*(M_4)$. (Here, q^μ are the coordinates of the position of a particle in space-time, and $p^\nu = \eta^{\nu\mu} p_\mu$ are the coordinates of a 4-vector of the momentum). When one transforms the coordinates $q'^\mu = q'^\mu(q^0, \ldots, q^3)$ the following transformation rule holds for the momentum:

$$p'_\mu(q^0, \ldots, q^3, p_0, \ldots, p_3) = \sum_{\nu=0}^{3} \frac{\partial q^\nu}{\partial q'^\mu} p_\nu \tag{4.2}$$

(q'^μ, p'_ν) is then the new coordinate system on $T^*(M_4)$ cannonically connected with the old one. Note that the momenta in the new coordinate system are linear functions of the momenta in the old coordinate system. In what follows we shall be mostly interested in the following special cases of (4.2), which generate the full conformal transformation group of M:

Translations:

$$q'^\mu = q^\mu + t^\mu, \qquad t \in R^4 \tag{4.3a}$$

Lorentz transformations:

$$q'^\mu = L_\nu{}^\mu q^\nu, \qquad L_\varphi{}^\mu L_\rho{}^\nu \eta_{\mu\nu} = \eta_{\varphi\rho}, \qquad \det[L_\nu{}^\mu] = +1, \operatorname{sgn} L_0{}^0 = +1 \tag{4.3b}$$

Dilatation:

$$q'^\mu = e^\alpha q^\mu, \alpha \in R^1 \tag{4.3c}$$

"Accelerations":

$$q'^\mu = \frac{q^\mu + q^2 c^\mu}{1 + 2qc + q^2 c^2}, \qquad c \in R^4 \tag{4.3d}$$

An arbitrary conformal transformation is the superposition of transformations of form (4.3). The "acceleration" transformation is not determined on the whole Minkowski space, as it does not make sense for $1 + 2qc + q^2 c^2 = 0$. Thus, the action of G_{conf} is not correctly defined on the whole of M_4. The formulas for the transformations of the momentum corresponding to (4.3) are

$$\text{Poincaré subgroup} \begin{cases} p'_\mu = p_\mu \qquad \text{or} \qquad p'^\mu = p^\mu & (4.4a) \\ p'_\mu = L_\mu^{-1\nu} p_\nu \qquad \text{or} \qquad p'^\mu = L_\nu{}^\mu p^\nu & (4.4b) \end{cases}$$

$$\text{dilatation } p'_\mu = e^{-\alpha} p^\mu \qquad \text{or} \qquad p'^\mu = e^{-\alpha} p^\mu \tag{4.4c}$$

and for the "accelerations"[1]

$$p'_\mu = \sum_{\nu=0}^{3} \frac{\partial q^\nu}{\partial q'^\mu} p_\nu = [\mathscr{H}(\delta_\mu{}^\nu + 2q_\mu c^\nu) - 2(c_\mu + c^2 q_\mu)(q^\nu + q^2 c^\nu)] p_\nu \tag{4.4d}$$

or

$$p'^\mu = \sum_{\nu=0}^{3} \frac{\partial q'^\mu}{\partial q^\nu} p^\nu = \mathscr{H}^{-2} [\mathscr{H}(\delta_\nu{}^\mu + 2q_\nu c^\mu) - 2(c_\nu + c^2 q_\nu)(q^\mu + q^2 c^\mu)] p^\nu$$

$$q_\mu = \eta_{\mu\nu} q^\nu, \qquad c_\nu = \eta_{\nu\varphi} c^\varphi \qquad \mathscr{H} = 1 + 2q \cdot c + q^2 c^2$$

The symplectic form σ is invariant with respect to arbitrary coordinate transformation $q'^\mu = q'^\mu(q^0, \ldots, q^3)$, thus, it is also conformally invariant. The relativistic mass is a scalar with respect to the Poincaré group but it changes under the dilatations and accelerations:

$$m' = e^{-\alpha} m \tag{4.5}$$

$$m' = \mathscr{H} m \tag{4.6}$$

Let us now denote $\mathscr{T}_4^+ = \{(q, p) \in T^*(M_4) : p^2 > 0 \text{ and } p^0 > 0\}$. The domain $\mathscr{T}_4^+$ is transformed into itself by the transformations (4.4). In what follows we shall be interested in particles for which $m^2 = p^2 > 0$ and $p^0 > 0$, and therefore further consideration will be restricted to the realizations of G_{conf} on $\mathscr{T}_4^+$.

5. *Physical Coordinates on (D, ω)*

The following fact takes place:

Statement. (D, ω) is isomorphic as a symplectic manifold to $(\mathscr{T}_4^+, 4\sigma|_{\mathscr{T}_4^+})$. The isomorphism which we shall construct below is interesting for the physical interpretation, for it allows one to identify the generalized disc with the phase space of a scalar positive mass particle. *The construction* consists in the transition to a new matrix representation of the group $SU(2, 2)$, which is connected with the old representation by the transformation $\mathrm{Ad}(d) : g \to \mathrm{Ad}(d)g = d^{-1}gd$, where $d = (1/\sqrt{2}) \left(\begin{smallmatrix} iE & iE \\ -E & E \end{smallmatrix}\right) \in U(4)$. Conditions for $g' = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) A, B, C,$ $D \in \mathrm{Mat}_2(\mathbb{C}^1)$ to be an element of the group $\mathrm{Ad}(d)[SU(2, 2) = d\, SU(2, 2) d^{-1}$ take the form

$$\begin{aligned} C^+A - A^+C &= 0 \\ A^+D - C^+B &= E, \\ D^+B - B^+D &= 0 \end{aligned} \qquad \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 1 \tag{5.1}$$

[1] Calculating (4.4d) we made use of the formula $\partial q^\nu / \partial q'^\mu = \mathscr{H}^2 \eta^{\nu\varphi} \eta_{\mu\rho} \partial q'^\rho / \partial q^\varphi$.

An arbitrary complex matrix $V \in \mathrm{Mat}_2(\mathbb{C}^1)$ may be uniquely decomposed into its Hermitian and anti-Hermitian parts $V = Q + iK$, where $G^+ = Q$ and $K^+ = K$. The domain $T_4^+ \overset{df}{=} \{V \in \mathrm{Mat}_2(\mathbb{C}^1): \det K > 0 \text{ and } \mathrm{Tr}\, K > 0\}$ is called the four-dimensional wedge. The matrix $d = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} = (1/\sqrt{2}) \begin{pmatrix} -iE & iE \\ E & E \end{pmatrix}$ induces the one-to-one holomorphic mapping $\mathscr{D}: D \to T_4^+$ given by the formula

$$V = D(z) \overset{df}{=} (d_{11}z + d_{12})(d_{21}z + d_{22})^{-1} = i(E - z)(E + z)^{-1} \tag{5.2}$$

We obtain the inverse transformation by taking the inverse matrix $d^{-1} = (1/\sqrt{2}) \begin{pmatrix} iE & E \\ -iE & E \end{pmatrix}$ instead of the matrix d

$$z = \mathscr{D}^{-1}(v) \overset{df}{=} (iv + E)(-iv + E)^{-1} \tag{5.3}$$

Both definitions are meaningful, since $\det(E + z) \neq 0$ for $z \in D$, and $\det(iv + E) \neq 0$ for $v \in T_4^+$. The $v = D(z)$ belongs to T_4^+ and $z = \mathscr{D}^{-1}(v) \in D$ since

$$\begin{aligned} \det K &= (1/16)|\det(V + iE)U|^2 \lambda_1 \lambda_2 \\ \mathrm{Tr}\, K &= (1/4)\, \mathrm{Tr} \left\{ [(V + iE)U]^+ (V + iE)U \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \right\} \end{aligned} \tag{5.4}$$

(we recall that λ_1, λ_2 are eigenvalues of the matrix $E - zz^+$). We shall define one more transformation, which maps the four-dimensional wedge onto the phase space $\mathscr{T}_4^+$ of a scalar particle with a positive mass. The transformation I (inversion), assigns to $Q + iK$ the point $x = x(q^0, \ldots, q^3, p^0, \ldots, p^3)$ such that

$$\begin{aligned} Q &= \sum_{\mu=0}^{3} q^\mu \sigma_\mu \\ \frac{1}{\det K} K &= \sum_{\mu=0}^{3} p^\mu \sigma_\mu \end{aligned} \tag{5.5}$$

The transformation I is a smooth one-to-one mapping of the four-dimensional wedge T_4^+ onto $\mathscr{T}_4^+$. The above facts may be expressed in form of the following commuting diagram:

$$\begin{tikzcd} D \arrow[r, "\mathscr{D}"] \arrow[d, "\sigma_D(g)"] & T_4^+ \arrow[r, "I"] \arrow[d, "\sigma_{T_4^+}(g')"] & \mathscr{T}_4^+ \arrow[d, "\sigma_{T_4^+}(g')"] \\ D \arrow[r, "\mathscr{D}"] & T_4^+ \arrow[r, "I"] & \mathscr{T}_4^+ \end{tikzcd} \tag{5.6}$$

Here, $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SU(2, 2)$ and $g' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = dgd^{-1} \in \mathrm{Ad}(d)[SU(2, 2)]$.

Thus, it is easy to see that the realization $\sigma_{T_4^+}$ of the group $\mathrm{Ad}(d)[SU(2, 2)]$ on T_4^+ is given by the formula

$$V' = \sigma_{T_4^+}(g', V = (AV + B)(CV + D)^{-1}, \qquad V = \mathscr{D}(z) \in T_4^+ \tag{5.7}$$

and the realization $\sigma_{\mathscr{T}_4^+}$ of the same group on $\mathscr{T}_4^+$ is given by

$$x' = \sigma_{\mathscr{T}_4^+}(g')x = (I \circ \sigma_{T_4^+}(g') \circ I^{-1})x, \qquad x \in \mathscr{T}_4^+ \tag{5.8}$$

The $\mathscr{D}^{-1}$ carries the symplectic structure from the generalized discus D to T_4^+. Namely

$$\overset{2}{\wedge}(\mathscr{D}^{-1})^*\omega = 8[(\vartheta - \bar{\vartheta})^2]^{-1}[2/(\vartheta - \bar{\vartheta})^2(\vartheta_\alpha - \bar{\vartheta}_\alpha)(\vartheta_\beta - \bar{\vartheta}_\beta) - \eta_{\alpha\beta}\, d\vartheta^\alpha \wedge d\bar{\vartheta}\beta \tag{5.9}$$

where ϑ are determined by the equation

$$V = \sum_{\mu=0}^{3} \vartheta^\mu \sigma_\mu$$

Similarly $(I \circ \mathscr{D})^{-1}$ carries the symplectic structure of D to $\mathscr{T}_4^+$. After a short calculation we get

$$\overset{2}{\wedge}(I \circ \mathscr{D}^{-1})^*\omega = 4\eta_{\mu\nu}\, dp^\nu \wedge dq^\nu = 4\sigma|_{\mathscr{T}_4^+} \tag{5.10}$$

From (5.10) it follows that $I \circ \mathscr{D} : D \to \mathscr{T}_4^+$ is an isomorphism between the symplectic manifold (D, ω) and $(\mathscr{T}_4^+, 4\sigma|_{\mathscr{T}_4^+})$. This completes the construction.

The mapping constructed above, $I \circ \mathscr{D}$, besides carrying the symplectic structure, transforms also the realization of the conformal group σ_D into the realization $\sigma_{\mathscr{T}_4^+}$, defined on $\mathscr{T}_4^+$. It is also interesting to note that the inverse mapping $(\mathscr{D} \circ I)^{-1}$ provides one of possible compactifications of the Minkowski space-time. In fact it maps M_4 into an open subset of the matrix group $U(2) \subset \partial D$. We shall discuss now some properties of the new realization of G_{conf}. This might be most conveniently done by specifying some subgroups of $\mathrm{Ad}(d)$ $(SU(2, 2))$, which poses a simple physical meaning. These subgroups can be distinguished by introducing the following basis in the algebra, $\mathrm{Ad}(d)[\mathscr{SU}(2, 2)] = d\mathscr{SU}(2, 2)\, d^{-1}$

$$P_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ 0 & 0 \end{pmatrix}, \quad Y_k = \begin{pmatrix} i\sigma_k & 0 \\ 0 & i\sigma_k \end{pmatrix}, \quad N_j = \begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix},$$

$$D = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}, \quad K_\nu = \begin{pmatrix} 0 & 0 \\ \sigma_\nu & 0 \end{pmatrix} \quad \mu, \nu = 0, 1, 2, 3, \qquad k, j = 1, 2, 3 \tag{5.11}$$

After performing the exponential transformation exp: $\mathrm{Ad}(d)[\mathcal{SU}(2, 2) \to \mathrm{Ad}(d)[SU(2, 2)]$ we find that

$$\exp t^{\mu}P_{\mu} = \begin{pmatrix} E & T \\ 0 & E \end{pmatrix}, \qquad T = t^{\mu}\sigma_{\mu}, \qquad T^{+} = T \tag{5.12a}$$

where P_{μ} are the generators of the translation subgroup

$$\exp x^{k}Y_{k} = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}, \qquad U = \exp ix^{k}\sigma_{k}, \qquad UU^{+} = E, \qquad \det U = 1 \tag{5.12b}$$

where I_k are generators of the rotation subgroup

$$\exp y^{j}N_{j} = \begin{pmatrix} R & 0 \\ 0 & R^{-1} \end{pmatrix}, \qquad R = \exp y^{k}\sigma_{k}, \qquad R^{+} = R, \qquad \det R = 1 \tag{5.12c}$$

where N_k are generators of the Lorentz rotation subgroup

$$\exp \alpha D = \begin{pmatrix} e^{\alpha}E & 0 \\ 0 & e^{-\alpha}E \end{pmatrix}, \qquad \alpha \in R^{1} \tag{5.12d}$$

where D is a generator of the dilatation subgroup

$$\exp c^{\nu}K_{\nu} = \begin{pmatrix} E & 0 \\ C & E \end{pmatrix}, \qquad C = c^{\nu}\sigma_{\nu}, \qquad C^{+} = c \tag{5.12e}$$

where K_{ν} are generators of the "acceleration" subgroup. The matrix $\left(\begin{smallmatrix} A & AT \\ 0 & (A^{+})^{-1} \end{smallmatrix}\right)$, where $A = RU$, corresponds to the general element (A, T) of the Poincaré group. The realization now takes the following form:

Translations:

$$\begin{aligned} q'^{\mu} &= q^{\mu} + t^{\mu} \\ p'^{\mu} &= p^{\mu} \end{aligned} \tag{5.13a}$$

Lorentz transformations:

$$\begin{aligned} q'^{\mu} &= L_{\nu}{}^{\mu} q^{\nu} \\ p'^{\mu} &= L_{\nu}{}^{\mu} p^{\nu} \end{aligned} \tag{5.13b}$$

where $L_{\nu}{}^{\mu}\sigma_{\mu} = A\sigma_{\nu}A^{+}$

Dilatation

$$\begin{aligned} q'^{\mu} &= e^{\alpha} q^{\mu} \\ p'^{\mu} &= e^{-\alpha} p^{\mu} \end{aligned} \tag{5.13c}$$

"Accelerations"

$$q'^{\mu} = \frac{[q^{\mu} + (q^2 - k^2)c^{\mu}]\,[1 + 2c\cdot q + c^2(q^2 - k^2)] + 2[c\cdot k + (q\cdot k)c^2]\,[k^{\mu} + 2(q\cdot k)c^{\mu}]}{[1 + 2c\cdot q + c^2(q^2 - k^2)]^2 + 4[c\cdot k + (q\cdot k)c^2]^2}$$

$$(5.13\text{d})$$

$$k'^{\mu} = \frac{-2[q^{\mu} + (q^2 - k^2)c^{\mu}]\,[c\cdot k + (q\cdot k)c^2] + [k^{\mu} + 2(q\cdot k)c^{\mu}]\,[1 + 2(c\cdot q) + c^2(q^2 - k^2)]}{[1 + 2c\cdot q + c^2(q^2 - k^2)]^2 + 4[c\cdot k + (q\cdot k)c^2]^2}$$

where

$$p'^{\mu} = k'^{\mu}/k'^2, \qquad p^{\mu} = k^{\mu}/k^2, \qquad K = k^{\mu}\sigma_{\mu}, \qquad c\cdot k = \eta_{\varphi\rho} c^{\varphi} k^{\rho}$$

The manner in which the specificated subgroups act on $\mathscr{T}_4^+$ justifies the terminology introduced. The Poincaré subgroup and dilatation subgroup act in the same way as they traditionally do on $T^*(M_4)$. The peculiarity lies in the form of the accelerations which now transform the momenta nonlinearly. For completeness we shall also specify the form of the two homomorphisms $d\sigma_{\mathscr{T}_4^+}$ and $\lambda_{\mathscr{T}_4^+}$ which appear in the following diagram

$$\begin{tikzcd} 0 \arrow[r] & R^1 \arrow[r, "\iota"] & C^{\infty}(\mathscr{T}_4^+, R^1) \arrow[r, "\gamma"] & H_0(\mathscr{T}_4^+) \arrow[r] & 0 \\ & & \mathrm{Ad}(d)[\mathscr{SU}(2,2)] \arrow[u, "\lambda_{\mathscr{T}_4^+}"] \arrow[ur, "d\sigma_{\mathscr{T}_4^+}"'] & & \end{tikzcd} \tag{5.14}$$

The values of these homomorphisms are

$$d\sigma_{\mathscr{T}_4^+}(P_{\alpha}) = \frac{\partial}{\partial q^{\alpha}} \tag{5.15a}$$

$$d\sigma_{\mathscr{T}_4^+}(L_{\alpha\beta}) = q_{\alpha}\frac{\partial}{\partial q^{\beta}} - q_{\beta}\frac{\partial}{\partial q^{\alpha}} + p_{\alpha}\frac{\partial}{\partial p^{\beta}} - p_{\beta}\frac{\partial}{\partial p^{\beta}} \tag{5.15b}$$

$$d\sigma_{\mathscr{T}_4^+}(D) = q^{\alpha}\frac{\partial}{\partial q^{\alpha}} - p^{\alpha}\frac{\partial}{\partial p^{\alpha}} \tag{5.15c}$$

$$d\sigma_{\mathscr{T}_4^+}(K_\beta) = \left(q^2 - \frac{1}{p^2}\right)\frac{\partial}{\partial q^\beta} + 2\left[\frac{p_\beta p^\gamma}{(p^2)^2} - q_\beta q^\gamma\right]\frac{\partial}{\partial q^\gamma} \qquad (5.15d)$$

$$+ 2[(q\cdot p)\delta_\beta{}^\gamma + q_\beta p^\gamma - p_\beta q^\gamma]\frac{\partial}{\partial p^\gamma}$$

Here we put $L_{01} = N_1, L_{02} = N_2, L_{03} = N_3, L_{23} = Y_1, L_{31} = Y_2, L_{12} = Y_3$ and $L_{\alpha\beta} = -L_{\beta\alpha}$. Subsequently we have

$$\lambda_{\mathscr{T}_4^+}(P_\alpha) = 4p_\alpha \qquad (5.16a)$$

$$\lambda_{\mathscr{T}_4^+}(L_{\alpha\beta}) = 4(q_\alpha p_\beta - q_\beta p_\alpha) \qquad (5.16b)$$

$$\lambda_{\mathscr{T}_4^+}(D) = 4q\cdot p \qquad (5.16c)$$

$$\lambda_{\mathscr{T}_4^+}(K_\beta) = 4[(q^2 - 1/p^2)p_\beta - 2(q\cdot p)q_\beta] \qquad (5.16d)$$

All the quantities quoted above are tensors with respect to the Lorentz group. The relativistic angular momentum $\lambda_{\mathscr{T}_4^+}(L_{\alpha\beta})$ and $\lambda_{\mathscr{T}_4^+}(D)$ are, in addition, invariant with respect to the dilatation.

6. *The Physical Interpretation*

In the preceding section it was shown that the symplectic manifold $(\mathscr{T}_4^+, 4\sigma|_{\mathscr{T}_4^+})$ can become a homogeneous space of G_{conf} in two ways: (a) as an open domain in the phase space $T^*(M_4)$ (with the natural symplectic structure and the standard action of G_{conf}); (b) as a symplectic space isomorphic to (D, ω). When one takes into account the symplectic structure alone, this distinction makes no difference. However, G-symplectic spaces corresponding to both cases (a) and (b) are essentially different. Thus, in case (a) the properties of conformal transformations of the phase space are "secondary": they are covariantly determined by the space-time conformal transformations, and, consistently, the momentum of a particle is always transformed in a linear way. In this case there is a mathematical difficulty caused by the impossibility of defining the transformation of "acceleration" on the whole space-time. Contrary to (a), the case (b) has an advantage of being mathematically correct since all conformal transformations including the "accelerations" are well defined. In this case, however, the conformal transformations in the phase space have a fundamental character and are not covariantly defined by the space-time transformations above. Inversely, the space-time transformations are "secondary" and they are determined by the phase space transformation. In order to compare both cases more explicitly, consider a massive ($m \gg 1$) and slowly moving ($|\mathbf{p}| \ll m$) physical object, (e.g., a nucleon, an atom). Then $k^\mu = p^\mu/p^2 = (p^0/m^2, \mathbf{p}/m^2)x = 0$. If we apply the zero approximation in (5.13) with respect to k^μ we get $k'^\mu = 0$ for $k^\mu = 0$ and we obtain for the space-time coordinates q^μ the transformation formulas, which coincide with the usual action of the conformal group on the space-time given by (4.3). If we consider the linear approximation in k^μ instead of the zero approximation, then after expanding (5.13d) into the Taylor series with respect to k^μ and neglecting the higher-order terms we obtain for the "acceleration"

$$q'^\mu \cong \frac{q^\mu + q^2 c^\mu}{1 + 2c\cdot q + c^2 q^2}$$

$$\begin{aligned} k'^\mu &\cong \sum_{\nu=0}^{3} \mathscr{H}^{-2}\,[\mathscr{H}(\delta_\nu{}^\mu + 2q_\nu c^\mu) - 2(c_\nu + c^2 q_\nu)(q^\mu + q^2 c^\mu)]\, k^\nu \\ &= \sum_{\nu=0}^{3} \frac{\partial q'^\mu}{\partial q^\nu} k^\nu \end{aligned} \tag{6.1}$$

and for the momentum

$$p'^\mu = \mathscr{H}^2 \sum_{\nu=0}^{3} \frac{\partial q'^\mu}{\partial q^\nu} p^\nu \quad \text{or} \quad p'_\mu = \sum_{\nu=0}^{3} \frac{\partial q^\nu}{\partial q'^\mu} p_\nu \tag{6.2}$$

As is seen, (6.2) coincide with (4.4d). The remaining transformations corresponding to the Poincaré and dilatation subgroups are linear, and so they coincide with their linear approximations. Hence, the case (a) corresponds to the linear approximation of the conformal transformations appearing in the case (b); the approximation $k^\mu \cong 0$ corresponds to a slowly moving massive particle, whereas the limit $m \to \infty$ corresponds to the zero approximation.

Now, one might think about a physical interpretation of the case (b) in the spirit of a "conformal relativity." In such a theory an "event" would be a point of the symplectic manifold $x = (q^\mu, p_\nu)$ and it could be interpreted as an act of creation (annihilation) of a particle with a momentum p^μ in space-time point q^μ. Consistently the "world line" of a particle would be one parameter family of "events" $x(t) = (q^\mu(t), p^\nu(t))$. Here the successive events on the "world line" would be interpreted as a kind of a continuous creation process. The above idea of the "world line" contains in a sense a more complete physical information than the orthodox one. It allows one to describe naturally the objects with changing mass (such as inelastically scattering particles): they are simply represented by these "world lines" that break the constraints condition $p^\mu(t)p_\mu(t) = m^2 = \text{const}$. The correspondence to the traditional events of special relativity is given by the family of frame-dependent projections $\Pi(s) : \mathscr{T}_4^+ \to M_4$ defined by $\Pi(s)x = (q^\mu)$, where (q^μ) are space-time coordinates of the event x in the "generalized inertial frames" s. The new character of the case (b) now finds its expression in the fact that, for two events x_1 and x_2 for which $\Pi(s)x_1 = \Pi(s)x_2$ in one reference frame s, one has, in general, $\Pi(s')x_1 \neq \Pi(s')x_2$. It thus follows that the "relativity" considered above provides a certain generalization of the very idea of the space-time. Its meaning can be best exhibited by comparison with the historically accepted models of space-time.

The Aristotelean Relativity.[2] The fact that two particles (with arbitrary

[2] By the Aristotelean time-space we understand $E^1 \times E^3$ where E^1 corresponds to time, E^3 corresponds to the space.

momenta) have been created (annihilated) in the same time (time coincidence) has an absolute sense. Similarly the fact that two particles have appeared in the same place (space coincidence) has an absolute sense.

The Galileo Relativity. The time coincidence of two acts of creation (annihilation) has an absolute sense but the space coincidence has only a relative meaning.

The Minkowski Relativity. Neither the time nor space coincidence have absolute sense. Still, the space-time coincidence has an absolute meaning independent of the type and momenta of the particles created (annihilated).

Conformal Relativity (*Case b*). Even the notion of the space-time coincidence loses its absolute meaning. Two particles with two different momenta, which, for one observer, have been created (annihilated) in the same space-time point, for another (accelerated) observer are created (annihilated) in two different space-time points. Two objects with different masses which, for one observer, have the coinciding world lines and therefore are always seen together can be seen separately by another observer. These effects disappear in the limit for the slowly moving ($|\mathbf{p}| \ll m$) and massive ($m \gg 1$) particles. They might suggest a possibility of a physical theory where the very concept of the space-time trajectory would lose its sense for high-energy particles when observed in high-acceleration regions.

The model of the conformal kinematics we have presented here was considered as the classical one. However, it is possible to perform the quantization making use of the quantization procedure given by Kostant (1970).

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Appendix

(1) *The Riemannian Structure Carried from D.* The symplectic form ω has been constructed as a counterpart of a metric form ds^2 on D. When carried on $\mathscr{T}_4^+$ it becomes

$$d\tilde{s}^2 = \overset{2}{\odot}[(I \circ \mathscr{D})^{-1}]^* ds^2 = 2[2p_\alpha p_\beta - p^2 \eta_{\alpha\beta}]\{dq^2 \odot dq^\beta + [1/(p^2)^2]\, dp^\alpha \odot dp^\beta\}$$

We find, that the form of the Laplace operator for the metric $d\tilde{s}^2$ is

$$\Delta = (-1/4)C_I$$

where C_I denotes the Casimir operator of the second order for the algebra

$d\sigma_{\mathscr{T}_4^+}\{\mathrm{Ad}(d)[SU(2,2)]\}$. This may be expressed by the formula

$$C_I = \tfrac{1}{2}\eta^{\alpha\gamma}\eta^{\beta\delta}\, d\sigma_{\mathscr{T}_4^+}(L_{\alpha\beta})\, d\sigma_{\mathscr{T}_4^+}(L_{\gamma\delta}) + \eta^{\alpha\beta}[d\sigma_{\mathscr{T}_4^+}(P_\alpha)\, d\sigma_{\mathscr{T}_4^+}(K_\beta) + d\sigma_{\mathscr{T}_4^+}(K_\alpha)\, d\sigma_{\mathscr{T}_4^+}(P_\beta)] - 2[d\sigma_{\mathscr{T}_4^+}(D)]^2$$

2. *(D, ω) as the Orbit in the Space Dual to the Lie Algebra $\mathscr{S}\mathscr{U}(2, 2)$.* As is known (see Kostant, 1970) all the symplectic G spaces of a Lie group G are exhausted by the orbits of the K representation in the space $\mathscr{G}^*$ dual to the Lie algebra $\mathscr{G}$ of the group G. The representation $K = \mathrm{Ad}^*$, where Ad denotes the adjoint representation of the group, and the asterisk denotes the conjugate representation. In the case of the group $SU(2, 2)$, by the introduction in $\mathscr{S}\mathscr{U}(2, 2)$ the scalar product $(x_1, x_2) \overset{df}{=} \mathrm{Tr}\, x_1 x_2,\ x_1, x_2 \in \mathscr{S}\mathscr{U}(2, 2)$. Thus (D, ω) is isomorphic to the orbit generated by the element $\lambda i\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}$, $\lambda \in R^1$.

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